

# Relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index

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Abstract. Poly-Bernoulli numbers  $B_n^{(k)} \in \mathbb{Q} (n \geq 0, k \in \mathbb{Z})$  are defined by Kaneko in 1997. Multi-Poly-Bernoulli numbers  $B_n^{(k_1, k_2, \dots, k_r)}$ , defined by using multiple polylogarithms, are generations of Kaneko's Poly-Bernoulli numbers  $B_n^{(k)}$ . We researched relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index in particular. In section 2, we introduce a identity for Multi-Poly-Bernoulli numbers of negative index which was proved by Kamano. In section 3, as main results, we introduce some relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index in particular.

## 1. INTRODUCTION

For any integer  $k$ , Kaneko[1] introduced Poly-Bernoulli numbers of index  $k$  by the following generating function:

$$\frac{Li_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

where  $Li_k(t)$  is the  $k$ -th polylogarithm defined by

$$Li_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}.$$

Since  $Li_1(t) = -\log(1 - t)$ , the number  $B_n^{(1)}$  is the ordinary  $n$ -th Bernoulli number  $B_n$ , which is defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

It is known that Poly-Bernoulli numbers of negative index are positive integers and we have a closed formula

$$B_n^{(-k)} = \sum_{j=0}^{\min(n, k)} (j!)^2 \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\}.$$

In particular, we have the following duality formula:

$$B_n^{(-k)} = B_k^{(-n)} \quad (k, n \geq 0).$$

Moreover, these numbers have combinatorial applications: see [2] and [3] for details.

As a generalization of Poly-Bernoulli numbers, Multi-Poly-Bernoulli numbers  $B_n^{(k_1, k_2, \dots, k_r)}$  are defined for integers  $k_1, \dots, k_r$  by the generating function

$$\frac{Li_{k_1, \dots, k_r}(1 - e^{-t})}{(1 - e^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^n}{n!},$$

where  $Li_{k_1, \dots, k_r}(t)$  is a multiple polylogarithm defined by

$$Li_{k_1, \dots, k_r}(t) = \sum_{0 < m_1 < \dots < m_r} \frac{t^{m_r}}{m_1^{k_1} \dots m_r^{k_r}}.$$

When  $r = 1$ , the number  $B_n^{(k)}$  is Poly-Bernoulli numbers. When  $r = 1$  and  $k_1 = 1$ , the number  $B_n^{(1)}$  is the classical Bernoulli numbers. It is also known that we have the following duality formula [4] for Multi-Poly Bernoulli numbers:

$$B_n^{(0, \dots, 0, -k)} = B_k^{(0, \dots, 0, -n)}.$$

## 2. RELATIONS OF MULTI-POLY-BERNOULLI NUMBERS OF NEGATIVE INDEX

In this section, we introduce a identity for Multi-Poly-Bernoulli numbers of negative index which was proved by Kamano [4]. If  $(k_1, \dots, k_r) \neq (0, \dots, 0)$ , then Multi-Poly-Bernoulli numbers of negative index  $B_n^{(-k_1, -k_2, \dots, -k_r)}$  have the following expression.

**Theorem 2.1.** Let  $r$  be a positive integer and let  $k_1, \dots, k_r$  be non-negative integers with  $(k_1, \dots, k_r) \neq (0, \dots, 0)$ . We put  $k := k_1 + \dots + k_r$ . Then the following identity holds:

$$B_n^{(-k_1, \dots, -k_r)} = \sum_{l=1}^k \alpha_l^{(k_1, \dots, k_r)} (l+r)^n \dots (A),$$

where  $\alpha_l^{(k_1, \dots, k_r)}$  ( $1 \leq l \leq k$ ) are integers depending only on  $k_1, \dots, k_r$ , and they are inductively determined by the following recurrence relations:

- (i)  $\alpha_l^{(k_1)} = (-1)^{l+k_1} l! \left\{ \begin{smallmatrix} k_1 \\ l \end{smallmatrix} \right\},$
- (ii)  $\alpha_l^{(k_1, \dots, k_{r-1}, 0)} = \alpha_l^{(k_1, \dots, k_{r-1})},$
- (iii)  $\alpha_l^{(k_1, \dots, k_{r-1}, k_r+1)} = (l+r-1) \alpha_{l-1}^{(k_1, \dots, k_r)} - l \alpha_l^{(k_1, \dots, k_r)}.$

Here we set

$$\alpha_0^{(k_1, \dots, k_r)} = \begin{cases} 1 & \text{if } (k_1, \dots, k_r) = (0, \dots, 0), \\ 0 & \text{otherwise,} \end{cases}$$

and  $\alpha_l^{(k_1, \dots, k_r)} = 0$  for  $l > k$ .

First we give recurrence relation [4] of Multi-Poly-Bernoulli numbers for the proof of the Theorem2.1.

**Lemma2.2.** For non-negative integers  $n, k_1, \dots, k_r$ , we have

$$B_n^{(-k_1, \dots, -k_{r-1}, -k_r-1)} = \sum_{m=0}^n \binom{n}{m} B_{m+1}^{(-k_1, \dots, -k_r)} + r B_n^{(-k_1, \dots, -k_r)} - B_{n+1}^{(-k_1, \dots, -k_r)}.$$

Theorem2.1 is proved by induction on  $r$ . The following lemma [4] says that Theorem2.1 holds for  $r = 1$ .

**Lemma2.3.** For  $n \geq 0$  and  $k \geq 1$ , we have

$$B_n^{(-k)} = \sum_{l=1}^k (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n.$$

We note here the Corollary2.4 has been proved by Hamahata and Masubuchi [5].

**Corollary2.4.** Let  $r$  be a positive integer and let  $n$  and  $k$  be non-negative integers. Then the following identities hold:

$$(1) \quad B_n^{\overbrace{(0, \dots, 0)}^r} = r^n,$$

$$(2) \quad (\text{duality}) \quad B_n^{(0, \dots, 0, -k)} = B_k^{(0, \dots, 0, -n)},$$

$$(3) \quad B_n^{(-k_1, \dots, -k_{r-1}, 0)} = \sum_{i=0}^n \binom{n}{i} B_i^{(-k_1, \dots, -k_{r-1})} \quad (r \geq 2),$$

$$(4) \quad \sum_{i=0}^k \binom{k}{i} B_n^{(-i, i-k)} p^i q^{k-i} = \sum_{i=0}^k \sum_{j=0}^n \binom{k}{i} \binom{n}{j} (p+q)^i q^{k-i} B_j^{(-i)} B_{n-j}^{(i-k)} \quad \text{where } p$$

and  $q$  are any real numbers.

We use the following generating function of Multi-Poly-Bernoulli numbers of negative index for the proof of Corollary2.4. This generating function is a natural generalization of the following function:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{1}{e^{-x} + e^{-y} - 1}.$$

**Theorem2.5.** The following identity holds:

$$\begin{aligned} & \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k_1, \dots, -k_r)} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \frac{t^n}{n!} \\ &= \frac{1}{(e^{-x_1-x_2-\dots-x_r} + e^{-t} - 1)(e^{-x_2-\dots-x_r} + e^{-t} - 1) \cdots (e^{-x_r} + e^{-t} - 1)}. \end{aligned}$$

We can express Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index in a sum of powers by using Theorem2.1. We give examples [4], [5] of Theorem2.1 for  $1 \leq r \leq 3$  and  $1 \leq k \leq 3$  (Table 1).

$r = 1$
$B_n^{(-1)} = 2^n$
$B_n^{(-2)} = -2^n + 2 \cdot 3^n$
$B_n^{(-3)} = 2^n - 6 \cdot 3^n + 6 \cdot 4^n$

$r = 2$
$B_n^{(0,-1)} = 2 \cdot 3^n$
$B_n^{(-1,0)} = 3^n$
$B_n^{(0,-2)} = -2 \cdot 3^n + 6 \cdot 4^n$
$B_n^{(-2,0)} = -3^n + 2 \cdot 4^n$
$B_n^{(0,-3)} = 2 \cdot 3^n - 18 \cdot 4^n + 24 \cdot 5^n$
$B_n^{(-1,-2)} = 3^n - 9 \cdot 4^n + 12 \cdot 5^n$
$B_n^{(-2,-1)} = 3^n - 7 \cdot 4^n + 8 \cdot 5^n$
$B_n^{(-3,0)} = 3^n - 6 \cdot 4^n + 6 \cdot 5^n$

$r = 3$
$B_n^{(0,0,-1)} = 3 \cdot 4^n$
$B_n^{(0,-1,0)} = 2 \cdot 4^n$
$B_n^{(-1,0,0)} = 4^n$
$B_n^{(0,0,-2)} = -3 \cdot 4^n + 12 \cdot 5^n$
$B_n^{(0,-2,0)} = -2 \cdot 4^n + 6 \cdot 5^n$
$B_n^{(-2,0,0)} = -4^n + 2 \cdot 5^n$
$B_n^{(0,-1,-1)} = -2 \cdot 4^n + 8 \cdot 5^n$
$B_n^{(-1,0,-1)} = -4^n + 4 \cdot 5^n$
$B_n^{(-1,-1,0)} = -4^n + 3 \cdot 5^n$
$B_n^{(0,0,-3)} = 3 \cdot 4^n - 36 \cdot 5^n + 60 \cdot 6^n$
$B_n^{(0,-3,0)} = 2 \cdot 4^n - 18 \cdot 5^n + 24 \cdot 6^n$
$B_n^{(-3,0,0)} = 4^n - 6 \cdot 5^n + 6 \cdot 6^n$
$B_n^{(0,-1,-2)} = 2 \cdot 4^n - 24 \cdot 5^n + 40 \cdot 6^n$
$B_n^{(0,-2,-1)} = 2 \cdot 4^n - 20 \cdot 5^n + 30 \cdot 6^n$
$B_n^{(-1,0,-2)} = 4^n - 12 \cdot 5^n + 20 \cdot 6^n$
$B_n^{(-1,-2,0)} = 4^n - 9 \cdot 5^n + 12 \cdot 6^n$
$B_n^{(-2,0,-1)} = 4^n - 8 \cdot 5^n + 10 \cdot 6^n$
$B_n^{(-2,-1,0)} = 4^n - 7 \cdot 5^n + 8 \cdot 6^n$
$B_n^{(-1,-1,-1)} = 4^n - 10 \cdot 5^n + 15 \cdot 6^n$

We found regularities from Table 1 and got the following relations. The proof uses Theorem2.1.

**Theorem2.6.** We have the following relations, and  $i$ -th component is  $-1$  and others are 0 in (3).

$$(1) B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})} = (r+1)^n = B_n^{(\overbrace{0, \dots, 0}^{r+1})}.$$

$$(2) B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)} = r(r+1)^n.$$

$$(3) B_n^{(\overbrace{0, \dots, 0}^r, -1, \overbrace{0, \dots, 0}^r)} = i(r+1)^n \quad (1 \leq i \leq r).$$

Proof. (1) We use Theorem 2.1(A) and (i),(ii). The second equality obtains from Corollary 1.4(1) .

$$\begin{aligned}
B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})} &= \sum_{l=1}^1 \alpha_l^{(1, \overbrace{0, \dots, 0}^{r-1})} (l+r)^n \\
&= \alpha_1^{(1, \overbrace{0, \dots, 0}^{r-1})} (1+r)^n \\
&= \alpha_1^{(1)} (1+r)^n.
\end{aligned}$$

Since  $\alpha_1^{(1)} = (-1)^{1+1} 1! \{1\}_1 = 1$ , we obtain  $B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})} = (r+1)^n = B_n^{(\overbrace{0, \dots, 0}^{r+1})}$ .

(2) We use Theorem 2.1(A) and (iii).

$$\begin{aligned}
B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)} &= \sum_{l=1}^1 \alpha_l^{(\overbrace{0, \dots, 0}^{r-1}, 1)} (l+r)^n \\
&= \alpha_1^{(\overbrace{0, \dots, 0}^{r-1}, 1)} (1+r)^n.
\end{aligned}$$

$$\begin{aligned}
\text{Here, } \alpha_1^{(\overbrace{0, \dots, 0}^{r-1}, 1)} &= (1+r-1) \alpha_0^{(\overbrace{0, \dots, 0}^r)} - 1 \cdot \alpha_1^{(\overbrace{0, \dots, 0}^r)} \\
&= r \alpha_0^{(\overbrace{0, \dots, 0}^r)} - \alpha_1^{(\overbrace{0, \dots, 0}^r)} = r.
\end{aligned}$$

$$\text{Thus, we obtain } B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)} = r(r+1)^n.$$

(3) We use Theorem 2.1(A) and (ii).

$$\begin{aligned}
B_n^{(\overbrace{0, \dots, 0}^r, -1, \overbrace{0, \dots, 0}^0)} &= \sum_{l=1}^1 \alpha_l^{(\overbrace{0, \dots, 0}^r, -1, \overbrace{0, \dots, 0}^0)} (l+r)^n \\
&= \alpha_1^{(\overbrace{0, \dots, 0}^r, -1, \overbrace{0, \dots, 0}^0)} (r+1)^n \\
&= \alpha_1^{(\overbrace{0, \dots, 0}^{i-1}, 1)} (r+1)^n.
\end{aligned}$$

Here from (2), we have  $B_n^{\overbrace{(0,\dots,0,-1)}^{i-1}} = i(i+1)^n$ . Moreover from Theorem 2.1(A), we have

$$\begin{aligned} B_n^{\overbrace{(0,\dots,0,-1)}^{i-1}} &= \sum_{l=1}^1 \alpha_l^{\overbrace{(0,\dots,0,1)}^{i-1}} (l+i)^n \\ &= \alpha_1^{\overbrace{(0,\dots,0,1)}^{i-1}} (i+1)^n. \end{aligned}$$

Since  $\alpha_1^{\overbrace{(0,\dots,0,1)}^{i-1}} = i$ , we obtain  $B_n^{\overbrace{(0,\dots,0,-1,0,\dots,0)}^r} = i(r+1)^n$ .

We have  $B_n^{\overbrace{(0,\dots,0,-1)}^{r-1}} = r B_n^{\overbrace{(-1,0,\dots,0)}^{r-1}}$  from (1) and (2). □

### 3. RELATIONS BETWEEN MULTI-POLY-BERNOULLI NUMBERS AND POLY-BERNOULLI NUMBERS OF NEGATIVE INDEX

In this section, we introduce relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index. Poly-Bernoulli numbers of negative index can express by using the Stirling numbers of the second kind  $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$  and Multi-Poly-Bernoulli numbers. Here Stirling numbers of the second kind are the number of ways to divide a set of  $n$  elements into  $m$  nonempty sets.

**Theorem 3.1.** Poly-Bernoulli numbers  $B_n^{(-k)}$  ( $n \geq 0, k \geq 1$ ) can express as follows;

$$\begin{aligned} (1) B_n^{(-k)} &= (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \left\{ \begin{smallmatrix} k \\ r+1 \end{smallmatrix} \right\} B_n^{\overbrace{(0,\dots,0,-1)}^r}. \\ (2) B_n^{(-k)} &= (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \left\{ \begin{smallmatrix} k \\ r+1 \end{smallmatrix} \right\} B_n^{\overbrace{(-1,0,\dots,0)}^r}. \\ (3) B_n^{(-k)} &= (-1)^{k-1} B_n^{(0,0)} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \frac{(r-i+2)!}{i} \left\{ \begin{smallmatrix} k \\ r-i+2 \end{smallmatrix} \right\} B_n^{\overbrace{(0,\dots,0,-1,0,\dots,0)}^{r-i+2}}. \end{aligned}$$

**Example 3.2.** We give examples of Theorem 3.1(1) and (2) for  $1 \leq k \leq 4$ .

$$\begin{aligned} (1) B_n^{(-1)} &= B_n^{(0,0)} \\ B_n^{(-2)} &= -B_n^{(0,0)} + B_n^{(0,-1)} \\ B_n^{(-3)} &= B_n^{(0,0)} - 3B_n^{(0,-1)} + 2B_n^{(0,0,-1)} \end{aligned}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 7B_n^{(0,-1)} - 12B_n^{(0,0,-1)} + 6B_n^{(0,0,0,-1)}$$

$$(2) B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(-1,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(-1,0)} + 6B_n^{(-1,0,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(-1,0)} - 36B_n^{(-1,0,0)} + 24B_n^{(-1,0,0,0)}$$

Proof of the Theorem 3.1.

(1) From Lemma 2.3, we have

$$B_n^{(-k)} = \sum_{l=1}^k (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n.$$

Here by putting  $l-1 = r$ , we obtain

$$\begin{aligned} B_n^{(-k)} &= \sum_{r=0}^{k-1} (-1)^{r+k-1} (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} (r+2)^n \\ &= (-1)^{k-1} 2^n + \sum_{r=1}^{k-1} (-1)^{r+k-1} (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} (r+2)^n. \end{aligned}$$

Since  $B_n^{(0,0)} = 2^n$  from Corollary 2.4(1), we have

$$B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1} r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} (r+1)(r+2)^n.$$

Here from Theorem 2.6(2), we have  $B_n^{\overbrace{(0,\dots,0,-1)}^{r-1}} = r(r+1)^n$ . Thus we have

$$B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1} r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} B_n^{\overbrace{(0,\dots,0,-1)}^r},$$

and we obtain the identity of (1).

(2) The proof of (2) uses the proof of (1). In the proof of (1), we have

$$B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1} (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} (r+2)^n.$$

Here from Theorem 2.6(1), we have  $B_n^{\overbrace{(-1,0,\dots,0)}^{r-1}} = (r+1)^n$ . Hence we have



$$B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1} (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})},$$

and we obtain the identity of (2).

(3) From Lemma 2.3, we have

$$B_n^{(-k)} = \sum_{l=1}^k (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n.$$

Here by putting  $l-1 = r-i+1$ , we obtain

$$\begin{aligned} B_n^{(-k)} &= \sum_{r=i-1}^{k+i-2} (-1)^{r-i+k} (r-i+2)! \left\{ \begin{matrix} k \\ r-i+2 \end{matrix} \right\} (r-i+3)^n \\ &= (-1)^{k-1} 2^n + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} (r-i+2)! \left\{ \begin{matrix} k \\ r-i+2 \end{matrix} \right\} \{(r-i+2)+1\}^n \\ &= (-1)^{k-1} B_n^{(0,0)} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} k \\ r-i+2 \end{matrix} \right\} i \{(r-i+2)+1\}^n. \end{aligned}$$

Here from Theorem 2.6(3), we have  $B_n^{(\overbrace{0, \dots, 0, -1, 0, \dots, 0}^r)} = i(r+1)^n$ . Thus we have

$$B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} k \\ r-i+2 \end{matrix} \right\} B_n^{(\overbrace{0, \dots, 0, -1, 0, \dots, 0}^{r-i+2})},$$

and we obtain the identity of (3).  $\square$

In the Theorem 3.1, when we replace  $r \rightarrow 2r$  and  $i \rightarrow r+1$ , we obtain the identity of (1). When we replace  $i \rightarrow 1$ , we obtain the identity of (2). Hence the identity of (3) is generalization of (1) and (2).

Futhermore we can also express Poly-Bernoulli numbers of negative index using the Stirling numbers of the first kind  $\left[ \begin{matrix} n \\ m \end{matrix} \right]$  and Multi-Poly-Bernoulli numbers. Here Stirling numbers of the first kind are the number of permutations of  $n$  letters (elements of the symmetric group of degree  $n$ ) that consist of  $m$  disjoint cycles.

**Corollary 3.3.** We have the following relations

$$(1) B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \left[ \begin{matrix} -r-1 \\ -k \end{matrix} \right] B_n^{(\overbrace{0, \dots, 0}^r, -1)}.$$

$$(2) B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \begin{bmatrix} -r-1 \\ -k \end{bmatrix} B_n^{(-1, \overbrace{0, \dots, 0}^r)}.$$

$$(3) B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \frac{(r-i+2)!}{i} \begin{bmatrix} -r+i-2 \\ -k \end{bmatrix} B_n^{\overbrace{(0, \dots, 0, -1, 0, \dots, 0)}^{r-i+2}}.$$

The proof of Corollary3.3 can be obtained from the following Lemma3.4 [1].

**Lemma3.4.** For any integers  $n$  and  $m$ , we have

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{Bmatrix} -m \\ -n \end{Bmatrix}.$$

Next we see the sum of coefficients on Multi-Poly-Bernoulli numbers of the identity which hold on Theorem3.1. Therefore we revisit Example3.2.

**Example3.2** (Example3.2 revisited).

We give examples of Theorem3.1(1) and (2) for  $1 \leq k \leq 4$ .

$$(1) B_n^{(-2)} = -B_n^{(0,0)} + B_n^{(0,-1)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 3B_n^{(0,-1)} + 2B_n^{(0,0,-1)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 7B_n^{(0,-1)} - 12B_n^{(0,0,-1)} + 6B_n^{(0,0,0,-1)}$$
  

$$(2) B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(-1,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(-1,0)} + 6B_n^{(-1,0,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(-1,0)} - 36B_n^{(-1,0,0)} + 24B_n^{(-1,0,0,0)}$$

In the case of (1), the sum of coefficients on Multi-Poly-Bernoulli numbers are 0 ( $k \geq 2$ ). In the case of (2), the sum of coefficients on Multi-Poly-Bernoulli numbers are 1 ( $k \geq 2$ ). From here we can be considered the following relations.

**Theorem3.5.** We have the following relations for  $k \geq 2$

$$(1) (-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \begin{Bmatrix} k \\ r+1 \end{Bmatrix} = 0.$$

$$(2) (-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \begin{Bmatrix} k \\ r+1 \end{Bmatrix} = 1.$$

$$(3) (-1)^{k-1} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} k \\ r-i+2 \end{matrix} \right\} = \begin{cases} 1 & (k: \text{ odd}) \\ \frac{2}{i} - 1 & (k: \text{ even}). \end{cases}$$

We regard the sums of coefficients as 1 for  $k = 1$ .

Proof. (1) We have

$$(-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} = (-1)^{k-1} \left( 1 + \sum_{r=1}^{k-1} (-1)^r r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \right).$$

Here,  $(-1)^r r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} = 1$  for  $r = 0$  and  $(-1)^r r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} = 0$  for  $r = k$ . Thus we have

$$\begin{aligned} (-1)^{k-1} \left( 1 + \sum_{r=1}^{k-1} (-1)^r r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \right) &= (-1)^{k-1} \left( 1 + \sum_{r=0}^k (-1)^r r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} - 1 \right) \\ &= (-1)^{k-1} \sum_{r=0}^k (-1)^r r! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \\ &= (-1)^{k-1} \sum_{r=0}^k (-1)^r \begin{bmatrix} r+1 \\ 1 \end{bmatrix} \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\}. \end{aligned}$$

Futhermore, since  $\sum_{l=0}^n (-1)^l \left\{ \begin{matrix} n \\ l \end{matrix} \right\} \begin{bmatrix} l \\ m \end{bmatrix} = (-1)^m \delta_{m,n} ([1])$ , we have

$$\begin{aligned} (-1)^k \sum_{r=0}^k (-1)^{r+1} \begin{bmatrix} r+1 \\ 1 \end{bmatrix} \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} &= (-1)^k \delta_{1,k} \\ &= 0, \end{aligned}$$

and we obtain the results.

(2) Considering in the same way with (1), we have

$$(-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} = (-1)^{k-1} \left( 1 + \sum_{r=1}^{k-1} (-1)^r (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \right)$$

$$\begin{aligned}
&= (-1)^{k-1} \sum_{r=0}^{k-1} (-1)^r (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \\
&= (-1)^{k-1} \sum_{r=0}^{k-1} (-1)^r \sum_{l=0}^{r+1} \begin{bmatrix} r+1 \\ l \end{bmatrix} \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \\
&= (-1)^{k-1} \sum_{r=0}^k (-1)^r \sum_{l=0}^k \begin{bmatrix} r+1 \\ l \end{bmatrix} \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \\
&= (-1)^k \sum_{l=0}^k \sum_{r=0}^k (-1)^{r+1} \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \begin{bmatrix} r+1 \\ l \end{bmatrix}.
\end{aligned}$$

Here we use the aforesaid formula again;  $\sum_{l=0}^n (-1)^l \left\{ \begin{matrix} n \\ l \end{matrix} \right\} \begin{bmatrix} l \\ m \end{bmatrix} = (-1)^m \delta_{m,n}$  ([1]).

Then we have

$$\begin{aligned}
(-1)^k \sum_{l=0}^k \sum_{r=0}^k (-1)^{r+1} \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} \begin{bmatrix} r+1 \\ l \end{bmatrix} &= (-1)^k \sum_{l=0}^k (-1)^l \delta_{l,k} \\
&= (-1)^k \cdot (-1)^k \\
&= 1.
\end{aligned}$$

(3) (i) If  $k$  is odd, we put  $k = 2m + 1$ . Then we have

$$\begin{aligned}
&(-1)^{2m} + \sum_{r=i}^{2m+i-1} (-1)^{r-i+2m+1} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} \\
&= 1 + \sum_{r=i}^{2m+i-1} (-1)^{r-i+2m+1} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\}.
\end{aligned}$$

Hence it suffices to show the following identity;

$$\begin{aligned}
&\sum_{r=i}^{2m+i-1} (-1)^{2m+r-i+1} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} = 0. \\
&\sum_{r=i}^{2m+i-1} (-1)^{2m+r-i+1} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} \\
&= \sum_{r=i-2}^{2m+i-1} (-1)^{2m+r-i+1} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} - \frac{1}{i} \left\{ \begin{matrix} 2m+1 \\ 1 \end{matrix} \right\} \\
&= \frac{1}{i} \sum_{r=i-2}^{2m+i-1} (-1)^{2m+r-i+1} (r-i+2)! \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} - \frac{1}{i}.
\end{aligned}$$

Here from Theorem 3.5(2), since  $(-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \left\{ \begin{matrix} k \\ r+1 \end{matrix} \right\} = 1$ , we put  $k = 2m + 1$ . Then we obtain

$$\begin{aligned}
(-1)^{2m} + \sum_{r=1}^{2m} (-1)^{r+2m} (r+1)! \left\{ \begin{matrix} 2m+1 \\ r+1 \end{matrix} \right\} &= 1 \\
\sum_{r=1}^{2m} (-1)^{r+2m} (r+1)! \left\{ \begin{matrix} 2m+1 \\ r+1 \end{matrix} \right\} &= 0 \\
\sum_{r=i}^{2m+i-1} (-1)^{2m+r-i+1} (r-i+2)! \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} &= 0 \\
\sum_{r=i-2}^{2m+i-1} (-1)^{2m+r-i+1} (r-i+2)! \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} - 1 &= 0 \\
\sum_{r=i-2}^{2m+i-1} (-1)^{2m+r-i+1} (r-i+2)! \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} &= 1.
\end{aligned}$$

Hence, we have  $\sum_{r=i}^{2m+i-1} (-1)^{2m+r-i+1} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m+1 \\ r-i+2 \end{matrix} \right\} = 0$  and, the sum of the coefficients are 1.

(ii) If  $k$  is even, we put  $k = 2m$ . Then we have

$$\begin{aligned}
&(-1)^{2m-1} + \sum_{r=i}^{2m+i-2} (-1)^{r-i+2m} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} \\
&= -1 + \sum_{r=i}^{2m+i-2} (-1)^{r-i+2m} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\}.
\end{aligned}$$

Hence, it suffices to show the following identity;

$$\begin{aligned}
&\sum_{r=i}^{2m+i-2} (-1)^{2m+r-i} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} = \frac{2}{i}. \\
&\sum_{r=i}^{2m+i-2} (-1)^{2m+r-i} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} \\
&= \sum_{r=i-2}^{2m+i-2} (-1)^{2m+r-i} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} + \frac{1}{i} \left\{ \begin{matrix} 2m \\ 1 \end{matrix} \right\}
\end{aligned}$$

$$= \frac{1}{i} \sum_{r=i-2}^{2m+i-2} (-1)^{2m+r-i} (r-i+2)! \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} + \frac{1}{i}.$$

Here from Theorem3.5(2), we put  $k = 2m$ . Then we have

$$\begin{aligned} (-1)^{2m-1} + \sum_{r=1}^{2m-1} (-1)^{r+2m+1} (r+1)! \left\{ \begin{matrix} 2m \\ r+1 \end{matrix} \right\} &= 1 \\ \sum_{r=1}^{2m-1} (-1)^{r+2m+1} (r+1)! \left\{ \begin{matrix} 2m \\ r+1 \end{matrix} \right\} &= 2 \\ \sum_{r=i}^{2m+i-2} (-1)^{2m+r-i} (r-i+2)! \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} &= 2 \\ \sum_{r=i-2}^{2m+i-2} (-1)^{2m+r-i} (r-i+2)! \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} + 1 &= 2 \\ \sum_{r=i-2}^{2m+i-2} (-1)^{2m+r-i} (r-i+2)! \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} &= 1. \end{aligned}$$

Hence, since  $\sum_{r=i}^{2m+i-2} (-1)^{2m+r-i} \frac{(r-i+2)!}{i} \left\{ \begin{matrix} 2m \\ r-i+2 \end{matrix} \right\} = \frac{2}{i}$ , the sum of coefficients are  $\frac{2}{i} - 1$ . □

We found that Poly-Bernoulli numbers of negative index can express using the Stirling numbers of the second kind  $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$  and the sum of Multi-Poly-Bernoulli numbers. This time, we introduce that special values of Multi-Poly-Bernoulli numbers which hold on Theorem2.6 can express by using the sum of Poly-Bernoulli numbers.

**Theorem3.6**( $r \geq 1$ ). We have the following relations

$$(1) B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})} = \frac{1}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)}.$$

$$(2) B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)} = \frac{1}{(r-1)!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)}.$$

$$(3) B_n^{(\overbrace{0, \dots, 0, -1, 0, \dots, 0}^r)} = \frac{i}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)}.$$

Proof of Theorem3.6.

$$\begin{aligned}
(1) \frac{1}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)} &= \frac{1}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} \sum_{l=1}^k (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \\
&= \frac{1}{r!} \sum_{k=0}^r \begin{bmatrix} r \\ k \end{bmatrix} \sum_{l=0}^k (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \\
&= \frac{1}{r!} \sum_{k=0}^r \begin{bmatrix} r \\ k \end{bmatrix} \sum_{l=0}^r (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \\
&= \frac{1}{r!} \sum_{l=0}^r (-1)^l l! (l+1)^n \sum_{k=l}^r (-1)^k \begin{bmatrix} r \\ k \end{bmatrix} \left\{ \begin{matrix} k \\ l \end{matrix} \right\} \\
&= \frac{1}{r!} \sum_{l=0}^r (-1)^l l! (l+1)^n (-1)^l \delta_{l,r} \\
&= \frac{1}{r!} \sum_{l=0}^r l! (l+1)^n \delta_{l,r} \\
&= \frac{1}{r!} \cdot r! (r+1)^n \\
&= (r+1)^n = B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})}.
\end{aligned}$$

(2) We consider in the same way as (1), and we obtain

$$\frac{1}{(r-1)!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)} = r(r+1)^n = B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)}.$$

(3) We consider in the same way as (1), and we obtain

$$\frac{i}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)} = i(r+1)^n = B_n^{(\overbrace{0, \dots, 0, -1, 0, \dots, 0}^r)}.$$

This completes the proof.  $\square$

The following Collorary3.7 can be obtained by using Lemma3.4 in the identity of Theorem3.6. Therefore we omit the proof.

**Collorary3.7**( $r \geq 1$ ). We have the following relations

$$\begin{aligned}
(1) B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})} &= \frac{1}{r!} \sum_{k=1}^r \begin{Bmatrix} -k \\ -r \end{Bmatrix} B_n^{(-k)}. \\
(2) B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)} &= \frac{1}{(r-1)!} \sum_{k=1}^r \begin{Bmatrix} -k \\ -r \end{Bmatrix} B_n^{(-k)}.
\end{aligned}$$

$$(3) B_n^{\overbrace{(0, \dots, 0, -1, 0, \dots, 0)}^r} = \frac{i}{r!} \sum_{k=1}^r \begin{Bmatrix} -k \\ -r \end{Bmatrix} B_n^{(-k)}.$$

**Example 3.8.**

We give examples of Theorem 3.6 for  $2 \leq r \leq 4$ .

$$B_n^{(-1,0)} = \frac{1}{2} B_n^{(-1)} + \frac{1}{2} B_n^{(-2)}$$

$$B_n^{(-1,0,0)} = \frac{1}{3} B_n^{(-1)} + \frac{1}{2} B_n^{(-2)} + \frac{1}{6} B_n^{(-3)}$$

$$B_n^{(-1,0,0,0)} = \frac{1}{4} B_n^{(-1)} + \frac{11}{24} B_n^{(-2)} + \frac{1}{4} B_n^{(-3)} + \frac{1}{24} B_n^{(-4)}$$

$$B_n^{(0,-1)} = B_n^{(-1)} + B_n^{(-2)}$$

$$B_n^{(0,0,-1)} = B_n^{(-1)} + \frac{3}{2} B_n^{(-2)} + \frac{1}{2} B_n^{(-3)}$$

$$B_n^{(0,0,0,-1)} = B_n^{(-1)} + \frac{11}{6} B_n^{(-2)} + B_n^{(-3)} + \frac{1}{6} B_n^{(-4)}$$

Moreover, we consider  $B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})}$  which are the generalizations of Theorem 3.6(1).

First, we consider in the case of  $m = 2$ , that is,  $B_n^{(-2, \overbrace{0, \dots, 0}^{r-1})}$ . We give examples for  $1 \leq r \leq 4$  and  $m = 2$ ;

$$B_n^{(-2)} = -2^n + 2 \cdot 3^n$$

$$B_n^{(-2,0)} = -3^n + 2 \cdot 4^n$$

$$B_n^{(-2,0,0)} = -4^n + 2 \cdot 5^n$$

$$B_n^{(-2,0,0,0)} = -5^n + 2 \cdot 6^n$$

...

We use Theorem 2.6(1) ( $B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})} = (r+1)^n$ ) and Theorem 3.6(1), and we have

$$B_n^{(-2, \overbrace{0, \dots, 0}^{r-1})} = \frac{2}{(r+1)!} \sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} B_n^{(-k)} - \frac{1}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)}.$$

We consider similarly in the case of  $m = 3$ , that is,  $B_n^{(-3, \overbrace{0, \dots, 0}^{r-1})}$ . Then we have



$$B_n^{(-3, \overbrace{0, \dots, 0}^{r-1})} = \frac{6}{(r+2)!} \sum_{k=1}^{r+2} \begin{bmatrix} r+2 \\ k \end{bmatrix} B_n^{(-k)} - \frac{6}{(r+1)!} \sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} B_n^{(-k)} \\ + \frac{1}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)}.$$

From here we can be considered the generalizations, that is,  $B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})}$  as follows.

**Theorem3.9.** We have the following relations on  $B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})}$

$$B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})} = \sum_{l=1}^m \frac{(-1)^{l+m} l! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\}}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)}.$$

Proof. Using Theorem2.6(1) and Theorem3.6(1), we have

$$\frac{1}{r!} \sum_{k=1}^r \begin{bmatrix} r \\ k \end{bmatrix} B_n^{(-k)} = (r+1)^n.$$

Here, we replace  $r \rightarrow r+l-1$ , and we have

$$\frac{1}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)} = (r+l)^n.$$

Hence we obtain

$$\sum_{l=1}^m \frac{(-1)^{l+m} l! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\}}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)} = \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} (r+l)^n.$$

Here the right hand of the last equality can be obtained by putting  $k_1 = m, k_2 = \dots = k_r = 0$  in Theorem2.1. Thus it equals to  $B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})}$  and we obtain Theorem3.9.  $\square$

By using Lemma3.4, we can also express Theorem3.9 by using the Stirling numbers of the second kind.

Next, we consider  $B_n^{(\overbrace{0, \dots, 0}^{r-1}, -m)}$  which are the generalizations of Theorem3.6(2) in the same way. We consider the small values on  $m$ , and we have

$$B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)} = \left\{ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right\} r(r+1)^n$$

$$\begin{aligned}
B_n^{\overbrace{(0,\dots,0)}^{r-1},-2} &= -\left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} r(r+1)^n + \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\} r(r+1)(r+2)^n \\
B_n^{\overbrace{(0,\dots,0)}^{r-1},-3} &= \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} r(r+1)^n - \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} r(r+1)(r+2)^n + \left\{ \begin{matrix} 3 \\ 3 \end{matrix} \right\} r(r+1)(r+2)(r+3)^n \\
&\dots
\end{aligned}$$

From here we can be considered the generalizations, that is,  $B_n^{\overbrace{(0,\dots,0)}^{r-1},-m}$  as follows.

**Theorem3.10.** We have the folowing relations on  $B_n^{\overbrace{(0,\dots,0)}^{r-1},-m}$

$$B_n^{\overbrace{(0,\dots,0)}^{r-1},-m} = \sum_{l=1}^m \frac{(-1)^{l+m} (r)_{l-1} \left\{ \begin{matrix} m \\ l \end{matrix} \right\}}{(r+l-2)!} \sum_{k=1}^{r+l-1} \left[ \begin{matrix} r+l-1 \\ k \end{matrix} \right] B_n^{(-k)}.$$

Here, we define  $(r)_l = r(r+1) \cdots (r+l-1)$  and  $(r)_0 = 1$ .

Proof. Using Theorem2.6(2) and Theorem3.6(2), we have

$$\frac{1}{(r-1)!} \sum_{k=1}^r \left[ \begin{matrix} r \\ k \end{matrix} \right] B_n^{(-k)} = r(r+1)^n.$$

Here we replace  $r \rightarrow r+l-1$ , and we have

$$\frac{1}{(r+l-2)!} \sum_{k=1}^{r+l-1} \left[ \begin{matrix} r+l-1 \\ k \end{matrix} \right] B_n^{(-k)} = (r+l-1)(r+l)^n.$$

Hence we obtain

$$\begin{aligned}
\sum_{l=1}^m \frac{(-1)^{l+m} (r)_{l-1} \left\{ \begin{matrix} m \\ l \end{matrix} \right\}}{(r+l-2)!} \sum_{k=1}^{r+l-1} \left[ \begin{matrix} r+l-1 \\ k \end{matrix} \right] B_n^{(-k)} &= \sum_{l=1}^m (-1)^{l+m} (r)_{l-1} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (r+l-1)(r+l)^n \\
&= \sum_{l=1}^m (-1)^{l+m} (r)_l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (r+l)^n.
\end{aligned}$$

Here from Theorem2.1, since we have  $B_n^{\overbrace{(0,\dots,0)}^{r-1},-m} = \sum_{l=1}^m \alpha_l^{(0,\dots,0,m)} (l+r)^n$ , we prove by induction on  $m$

$$\text{and } l \text{ that we have } \alpha_l^{\overbrace{(0,\dots,0)}^{r-1},m} = (-1)^{l+m} (r)_l \left\{ \begin{matrix} m \\ l \end{matrix} \right\}.$$

First, we prove that we have  $\alpha_1^{(0,\dots,0,m)} = (-1)^{m+1} r \cdots (B)$ .

From Theorem 2.1(iii), since we have  $\alpha_1^{(0,\dots,0,m)} = r\alpha_0^{(0,\dots,0,m-1)} - \alpha_1^{(0,\dots,0,m-1)}$ , we have (B) for  $m = 1$ .

We assume that we have  $\alpha_1^{(0,\dots,0,k)} = (-1)^{k+1}r$  for  $m = k$  ( $k \geq 1$ ).

For  $m = k + 1$ , we have

$$\begin{aligned}\alpha_1^{(0,\dots,0,k+1)} &= r\alpha_0^{(0,\dots,0,k)} - \alpha_1^{(0,\dots,0,k)} \\ &= -\alpha_1^{(0,\dots,0,k)} \\ &= -(-1)^{k+1}r \\ &= (-1)^{k+2}r.\end{aligned}$$

Since this shows that (B) is true for  $m = k + 1$ , we have (B) for all integers  $m$ .

Next, we prove that we have  $\alpha_l^{(0,\dots,0,m)} = (-1)^{l+m}(r)_l \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} \cdots (C)$ .

From (B), we have (C) for  $m = 1$ .

We assume that we have  $\alpha_k^{(0,\dots,0,m)} = (-1)^{k+m}(r)_k \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$  for  $l = k$  ( $k \geq 1$ ).

For  $l = k + 1$ , we have

$$\begin{aligned}\alpha_{k+1}^{(0,\dots,0,m)} &= (k+r)\alpha_k^{(0,\dots,0,m-1)} - (k+1)\alpha_{k+1}^{(0,\dots,0,m-1)} \\ &= (k+r)\alpha_k^{(0,\dots,0,m-1)} - (k+1)\{(k+r)\alpha_k^{(0,\dots,0,m-2)} - (k+1)\alpha_{k+1}^{(0,\dots,0,m-2)}\} \\ &= (k+r)\alpha_k^{(0,\dots,0,m-1)} - (k+1)(k+r)\alpha_k^{(0,\dots,0,m-2)} + (k+1)^2\alpha_{k+1}^{(0,\dots,0,m-2)} \\ &= (-1)^{k+m-1}(r)_k(k+r)\left\{ \begin{smallmatrix} m-1 \\ k \end{smallmatrix} \right\} - (-1)^{k+m-2}(r)_k(k+1)(k+r)\left\{ \begin{smallmatrix} m-2 \\ k \end{smallmatrix} \right\} \\ &\quad + (k+1)^2\alpha_{k+1}^{(0,\dots,0,m-2)} \\ &\quad \dots \\ &= (-1)^{k+m-1}(r)_{k+1}\left[\left\{ \begin{smallmatrix} m-1 \\ k \end{smallmatrix} \right\} + (k+1)\left\{ \begin{smallmatrix} m-2 \\ k \end{smallmatrix} \right\} + \cdots + (k+1)^{m-k-1}\left\{ \begin{smallmatrix} m-(m-k) \\ k \end{smallmatrix} \right\}\right] \\ &\quad \times \left\{ \begin{smallmatrix} m-(m-k) \\ k \end{smallmatrix} \right\}.\end{aligned}$$

Since the Stirling numbers of the second kind satisfy the recurrence formula

$$\left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} + (m+1)\left\{ \begin{smallmatrix} n \\ m+1 \end{smallmatrix} \right\},$$

We have  $\left\{ \begin{smallmatrix} m-1 \\ k \end{smallmatrix} \right\} + (k+1)\left\{ \begin{smallmatrix} m-2 \\ k \end{smallmatrix} \right\} + (k+1)^2\left\{ \begin{smallmatrix} m-3 \\ k \end{smallmatrix} \right\} + \cdots + (k+1)^{m-k-1}\left\{ \begin{smallmatrix} m-(m-k) \\ k \end{smallmatrix} \right\}$

$$= \sum_{i=1}^{m-k} (k+1)^{i-1} \left\{ \begin{smallmatrix} m-i \\ k \end{smallmatrix} \right\}$$

$$\begin{aligned}
&= \sum_{i=1}^{m-k} (k+1)^{i-1} \left( \left\{ \begin{matrix} m-i+1 \\ k+1 \end{matrix} \right\} - (k+1) \left\{ \begin{matrix} m-i \\ k+1 \end{matrix} \right\} \right) \\
&= \sum_{i=1}^{m-k} \left[ (k+1)^{i-1} \left\{ \begin{matrix} m-i+1 \\ k+1 \end{matrix} \right\} - (k+1)^i \left\{ \begin{matrix} m-i \\ k+1 \end{matrix} \right\} \right] \\
&= \left\{ \begin{matrix} m \\ k+1 \end{matrix} \right\} - (k+1) \left\{ \begin{matrix} m-1 \\ k+1 \end{matrix} \right\} + (k+1) \left\{ \begin{matrix} m-1 \\ k+1 \end{matrix} \right\} - (k+1)^2 \left\{ \begin{matrix} m-2 \\ k+1 \end{matrix} \right\} + \dots \\
&\quad + (k+1)^{m-k-1} \left\{ \begin{matrix} k+1 \\ k+1 \end{matrix} \right\} - (k+1)^{m-k} \left\{ \begin{matrix} k \\ k+1 \end{matrix} \right\} \\
&= \left\{ \begin{matrix} m \\ k+1 \end{matrix} \right\}.
\end{aligned}$$

Therefore we have  $\alpha_{k+1}^{(0,\dots,0,m)} = (-1)^{k+m-1} (r)_{k+1} \left\{ \begin{matrix} m \\ k+1 \end{matrix} \right\}$  and this shows that (C) is true for  $l = k + 1$ .

Hence, we have (C) for all integers  $l, m$ , and this completes the proof.  $\square$

By using Lemma3.4, we can also express Theorem3.10 by using the Stirling numbers of the second kind.

Finally, we consider the case of  $B_n^{\overbrace{(0,\dots,0,-m,0,\dots,0)}^r}$  (where  $i$ -th component is  $-m$  and others are 0) which are the extension of Theorem3.9 and Theorem3.10. For example, we fluctuate the value of 2-th. Then we obtain

$$\begin{aligned}
B_n^{\overbrace{(0,-1,0,\dots,0)}^r} &= 2(r+1)^n \\
B_n^{\overbrace{(0,-2,0,\dots,0)}^r} &= -2(r+1)^n + 6(r+2)^n \\
B_n^{\overbrace{(0,-3,0,\dots,0)}^r} &= 2(r+1)^n - 6(r+2)^n + 12(r+3)^n \\
B_n^{\overbrace{(0,-4,0,\dots,0)}^r} &= -2(r+1)^n + 6(r+2)^n - 12(r+3)^n + 24(r+4)^n \\
&\dots
\end{aligned}$$

By using this relations and the recurrence formula on Theorem2.1, we can be considered the following relations.

**Theorem3.11.** We have the following relations on  $B_n^{\overbrace{(0,\dots,0,-m,0,\dots,0)}^r}$ , and  $i$ -th component is  $-m$  and others are 0;

$$B_n^{\overbrace{(0, \dots, 0, -m, 0, \dots, 0)}^r} = \sum_{l=1}^m \frac{(-1)^{l-m} (i)_l \{m\}_l}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)}.$$

Proof. In the proof of Theorem3.9, we have

$$\frac{1}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)} = (r+l)^n.$$

Futhermore, using Theorem2.1 and  $\alpha_l^{\overbrace{(0, \dots, 0, m)}^{r-1}} = (-1)^{l+m} (r)_l \{m\}_l$  in the proof of Theorem3.10, we have

$$\begin{aligned} \sum_{l=1}^m \frac{(-1)^{l-m} (i)_l \{m\}_l}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)} &= \sum_{l=1}^m (-1)^{l-m} (i)_l \{m\}_l (r+l)^n \\ &= \sum_{l=1}^m \alpha_l^{\overbrace{(0, \dots, 0, m)}^{i-1}} (r+l)^n \\ &= \sum_{l=1}^m \alpha_l^{\overbrace{(0, \dots, 0, m, 0, \dots, 0)}^{i-1}} (r+l)^n \\ &= B_n^{\overbrace{(0, \dots, 0, -m, 0, \dots, 0)}^r}. \end{aligned}$$

Hence we obtain

$$B_n^{\overbrace{(0, \dots, 0, -m, 0, \dots, 0)}^r} = \sum_{l=1}^m \frac{(-1)^{l+m} (i)_l \{m\}_l}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)},$$

and This completes the proof.  $\square$

By using Lemma3.4, we can also express Theorem3.11 by using the Stirling numbers of the second kind.

If we put  $i = 1$ ,  $i = r$  in Theorem3.11, we obtain the following

$$\begin{aligned} B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})} &= \sum_{l=1}^m \frac{(-1)^{l+m} l! \{m\}_l}{(r+l-1)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)}, \\ B_n^{\overbrace{(0, \dots, 0, -m)}^{r-1}} &= \sum_{l=1}^m \frac{(-1)^{l+m} (r)_{l-1} \{m\}_l}{(r+l-2)!} \sum_{k=1}^{r+l-1} \begin{bmatrix} r+l-1 \\ k \end{bmatrix} B_n^{(-k)}. \end{aligned}$$

Hence we find that Theorem3.11 is the extension of Theorem3.9 and Theorem3.10.

Here we represent  $B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})}$ ,  $B_n^{(\overbrace{0, \dots, 0}^{r-1}, -m)}$ ,  $B_n^{(\overbrace{0, \dots, 0, -m, 0, \dots, 0}^r)}$  in the form of powers on  $r + l$  ( $1 \leq l \leq m$ ), and we see the sum of coefficients. For example, we put  $m = 1$ . Then we have the following relations from Theorem 2.6:

$$\begin{aligned} B_n^{(-1, \overbrace{0, \dots, 0}^{r-1})} &= (r+1)^n, \\ B_n^{(\overbrace{0, \dots, 0}^{r-1}, -1)} &= r(r+1)^n, \\ B_n^{(\overbrace{0, \dots, 0, -1, 0, \dots, 0}^r)} &= i(r+1)^n. \end{aligned}$$

Hence each coefficients are 1,  $r$ , and  $i$ . From this results, we can be considered the following.

**Theorem 3.12.** We have the following relations on the sum of coefficients

- (1) The sum of coefficients on  $B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})}$  are 1.
- (2) The sum of coefficients on  $B_n^{(\overbrace{0, \dots, 0}^{r-1}, -m)}$  are  $r^m$ .
- (3) The sum of coefficients on  $B_n^{(\overbrace{0, \dots, 0, -m, 0, \dots, 0}^r)}$  are  $i^m$ .

Proof. (1) In the proof of Theorem 3.9, we have

$$B_n^{(-m, \overbrace{0, \dots, 0}^{r-1})} = \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (r+l)^n.$$

Hence it suffices to show that  $\sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} = 1$ . We have

$$\begin{aligned} \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} &= (-1)^m \sum_{l=1}^m (-1)^l \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \\ &= (-1)^m \sum_{l=0}^m (-1)^l \sum_{k=0}^m \begin{bmatrix} l \\ k \end{bmatrix} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \\ &= (-1)^m \sum_{k=0}^m \sum_{l=0}^m (-1)^l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \begin{bmatrix} l \\ k \end{bmatrix}. \end{aligned}$$

Here since  $\sum_{l=0}^n (-1)^l \left\{ \begin{matrix} n \\ l \end{matrix} \right\} \left[ \begin{matrix} l \\ m \end{matrix} \right] = (-1)^m \delta_{m,n} ([1])$ , we obtain

$$\begin{aligned} \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} &= (-1)^m \sum_{k=0}^m (-1)^k \delta_{k,m} \\ &= (-1)^m (-1)^m \delta_{m,m} \\ &= 1. \end{aligned}$$

(2) In the proof of Theorem3.10, we have

$$B_n^{\overbrace{(0, \dots, 0, -m)}^{r-1}} = \sum_{l=1}^m (-1)^{l+m} (r)_l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (r+l)^n.$$

Hence it suffices to show that  $\sum_{l=1}^m (-1)^{l+m} (r)_l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} = r^m$ .

$$\text{Since } x^n = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n-k} (x)_k \quad (n \geq 0) \quad ([6]), \quad \sum_{l=1}^m (-1)^{l+m} (r)_l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} = r^m.$$

(3) In the proof of Theorem3.11, we have

$$B_n^{\overbrace{(0, \dots, 0, -m, 0, \dots, 0)}^r} = \sum_{l=1}^m (-1)^{l+m} (i)_l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (r+l)^n.$$

Hence it suffices to show that  $\sum_{l=1}^m (-1)^{l+m} (i)_l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} = i^m$ . This identity can be obtained by putting  $r = i$  in (2), and we obtain the result.  $\square$

For  $m = 1$ , since each coefficients are 1,  $r$ , and  $i$ , we find that Theorem3.12 is the generalizations.

We introduced several relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers up to here. We obtained Theorem3.1 by fluctuating values of  $r$  which represent numbers of 0. Here we consider that fluctuating values of  $m$  which represent numbers except for 0.

**Theorem3.13.** We have the following relations

$$(1) \quad B_n^{(-k)} = \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k}{m} B_n^{(-m,0)} \quad (k \geq 1).$$

Where the sum of coefficients on  $B_n^{(-m,0)}$  are 1.

$$(2) \quad B_n^{(-k)} = \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} B_n^{(0,-m)} \quad (k \geq 1).$$

Where the sum of coefficients on  $B_n^{(0,-m)}$  are 0 for  $k \geq 2$ . we regard the sum on the right hand as 1 if  $k = 1$ .

$$\begin{aligned}
\text{Proof. (1) R.H.S.} &= \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k}{m} \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (l+2)^n \\
&= \sum_{m=0}^{k-1} \binom{k}{m} \sum_{l=1}^m (-1)^{l+1+k} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (l+2)^n \\
&= \sum_{m=0}^{k-1} \binom{k}{m} \sum_{l=0}^m (-1)^{l+1+k} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (l+2)^n \\
&= \sum_{m=0}^{k-1} \binom{k}{m} \sum_{l=0}^{k-1} (-1)^{l+1+k} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (l+2)^n \\
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} l! (l+2)^n \sum_{m=0}^{k-1} \binom{k}{m} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \\
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} l! (l+2)^n \left[ \sum_{m=0}^k \binom{k}{m} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} - \left\{ \begin{matrix} k \\ l \end{matrix} \right\} \right] \\
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} l! (l+2)^n \left[ \left\{ \begin{matrix} k+1 \\ l+1 \end{matrix} \right\} - \left\{ \begin{matrix} k \\ l \end{matrix} \right\} \right] \\
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} l! (l+2)^n (l+1) \left\{ \begin{matrix} k \\ l+1 \end{matrix} \right\} \\
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+1)! \left\{ \begin{matrix} k \\ l+1 \end{matrix} \right\} (l+2)^n \\
&= \sum_{l=1}^k (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \\
&= B_n^{(-k)}.
\end{aligned}$$

Hence we obtain the result. Furthermore, the sum of coefficients on  $B_n^{(-m,0)}$  are

$$\sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k}{m} = \sum_{m=0}^k (-1)^{k-m-1} \binom{k}{m} + 1$$



$$\begin{aligned}
&= - \sum_{m=0}^k (-1)^{k-m} \binom{k}{m} + 1 \\
&= -(1-1)^k + 1 \\
&= 1.
\end{aligned}$$

This completes the proof and we obtain (1).

(2) We put  $r = 2$  in Theorem 3.10, and we use Theorem 3.6 and Theorem 2.6. Then we have

$$\begin{aligned}
B_n^{(0,-m)} &= \sum_{l=1}^m \frac{(-1)^{l+m} (2)_{l-1} \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\}}{l!} \sum_{k=1}^{l+1} \left[ \begin{smallmatrix} l+1 \\ k \end{smallmatrix} \right] B_n^{(-k)} \\
&= \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} \frac{1}{l!} \sum_{k=1}^{l+1} \left[ \begin{smallmatrix} l+1 \\ k \end{smallmatrix} \right] B_n^{(-k)} \\
&= \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} B_n^{(\overbrace{0, \dots, 0}^l, -1)} \\
&= \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} (l+1)(l+2)^n.
\end{aligned}$$

Hence we substitute this identity on the right of Theorem 3.13, and we have

$$\begin{aligned}
\text{R.H.S.} &= \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} (l+1)(l+2)^n \\
&= \sum_{m=0}^{k-1} \binom{k-1}{m} \sum_{l=1}^m (-1)^{l+1+k} (l+1)! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} (l+2)^n \\
&= \sum_{m=0}^{k-1} \binom{k-1}{m} \sum_{l=0}^m (-1)^{l+1+k} (l+1)! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} (l+2)^n \\
&= \sum_{m=0}^{k-1} \binom{k-1}{m} \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+1)! \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} (l+2)^n \\
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+1)! (l+2)^n \sum_{m=0}^{k-1} \binom{k-1}{m} \left\{ \begin{smallmatrix} m \\ l \end{smallmatrix} \right\} \\
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+1)! (l+2)^n \left\{ \begin{smallmatrix} k \\ l+1 \end{smallmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+1)! \left\{ \begin{matrix} k \\ l+1 \end{matrix} \right\} (l+2)^n \\
&= \sum_{l=1}^k (-1)^{l+k} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \\
&= B_n^{(-k)}.
\end{aligned}$$

Therefore we obtain the result. Furthermore, the sum of coefficients on  $B_n^{(0,-m)}$  are

$$\sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} = (1-1)^{k-1} = \begin{cases} 1 & (k=1) \\ 0 & (k \geq 2). \end{cases}$$

This completes the proof and we obtain (2).  $\square$

We extend Theorem 3.13, and we can write  $B_n^{(-k)}$  by using the sum of  $B_n^{\overbrace{(0,\dots,0)}^{l+1}}$  ( $l \geq 1$ ) and them of  $B_n^{\overbrace{(-m,0,\dots,0)}^r}$ , or  $B_n^{\overbrace{(0,\dots,0,-m)}^r}$ . First, we write  $B_n^{(-k)}$  by using the sum of  $B_n^{\overbrace{(0,\dots,0)}^{l+1}}$  ( $l \geq 1$ ) and them of  $B_n^{\overbrace{(-m,0,\dots,0)}^r}$ .

**Theorem 3.14.** We have the following relations

$$B_n^{(-k)} = \sum_{l=1}^r (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} B_n^{\overbrace{(0,\dots,0)}^{l+1}} + \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \left\{ \begin{matrix} k-m \\ r \end{matrix} \right\} B_n^{\overbrace{(-m,0,\dots,0)}^r}.$$

$$\text{Proof. (1)} \quad \sum_{l=1}^r (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} B_n^{\overbrace{(0,\dots,0)}^{l+1}} = \sum_{l=1}^r (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n.$$

Furthermore, we have

$$\begin{aligned}
&\sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \left\{ \begin{matrix} k-m \\ r \end{matrix} \right\} B_n^{\overbrace{(-m,0,\dots,0)}^r} \\
&= \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \left\{ \begin{matrix} k-m \\ r \end{matrix} \right\} \sum_{l=1}^m (-1)^{l+m} l! \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (l+r+1)^n \\
&= \sum_{m=r+1}^k (-1)^{k-m} \binom{k}{m-r} r! \left\{ \begin{matrix} k-m+r \\ r \end{matrix} \right\} \sum_{l=1}^{m-r} (-1)^{l+m-r} l! \left\{ \begin{matrix} m-r \\ l \end{matrix} \right\} (l+r+1)^n \\
&= \sum_{m=r+1}^k \binom{k}{m-r} r! \left\{ \begin{matrix} k-m+r \\ r \end{matrix} \right\} \sum_{l=1}^{k-1} (-1)^{k-l-r} l! \left\{ \begin{matrix} m-r \\ l \end{matrix} \right\} (l+r+1)^n
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=r+1}^k \binom{k}{m-r} r! \left\{ \begin{matrix} k-m+r \\ r \end{matrix} \right\} \sum_{l=1}^k (-1)^{k-l-r} l! \left\{ \begin{matrix} m-r \\ l \end{matrix} \right\} (l+r+1)^n \\
&= \sum_{l=1}^k (-1)^{k-l-r} r! l! (l+r+1)^n \sum_{m=r+1}^k \left\{ \begin{matrix} m-r \\ l \end{matrix} \right\} \left\{ \begin{matrix} k-m+r \\ r \end{matrix} \right\} \binom{k}{m-r} \\
&= \sum_{l=1}^k (-1)^{k-l-r} r! l! (l+r+1)^n \left\{ \begin{matrix} k \\ l+r \end{matrix} \right\} \binom{l+r}{l} \\
&= \sum_{l=1}^{k-r} (-1)^{k-l-r} r! l! (l+r+1)^n \left\{ \begin{matrix} k \\ l+r \end{matrix} \right\} \binom{l+r}{l} \\
&= \sum_{l=1}^{k-r} (-1)^{k-l-r} \left\{ \begin{matrix} k \\ l+r \end{matrix} \right\} (l+r+1)^n r! l! \frac{(l+r)!}{l! r!} \\
&= \sum_{l=1}^{k-r} (-1)^{k-l-r} (l+r)! \left\{ \begin{matrix} k \\ l+r \end{matrix} \right\} (l+r+1)^n \\
&= \sum_{l=r+1}^k (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \quad (l \rightarrow l-r).
\end{aligned}$$

Hence the right hand of Theorem3.14 can be expressed as follows, and we obtain the result.

$$\begin{aligned}
\text{R.H.S.} &= \sum_{l=1}^r (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n + \sum_{l=r+1}^k (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \\
&= \sum_{l=1}^k (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} (l+1)^n \\
&= B_n^{(-k)}.
\end{aligned}$$

□

### Example3.15.

We give examples of Theorem3.14 for  $1 \leq r \leq 2, 1 \leq k \leq 6$ .

(i) For  $r = 1$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(-1,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 3B_n^{(-1,0)} + 3B_n^{(-2,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 4B_n^{(-1,0)} - 6B_n^{(-2,0)} + 4B_n^{(-3,0)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 5B_n^{(-1,0)} + 10B_n^{(-2,0)} - 10B_n^{(-3,0)} + 5B_n^{(-4,0)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 6B_n^{(-1,0)} - 15B_n^{(-2,0)} + 20B_n^{(-3,0)} - 15B_n^{(-4,0)} + 6B_n^{(-5,0)}$$

(ii) For  $r = 2$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 6B_n^{(-1,0,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 24B_n^{(-1,0,0)} + 12B_n^{(-2,0,0)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 70B_n^{(-1,0,0)} - 60B_n^{(-2,0,0)} + 20B_n^{(-3,0,0)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 180B_n^{(-1,0,0)} + 210B_n^{(-2,0,0)} - 120B_n^{(-3,0,0)} + 30B_n^{(-4,0,0)}$$

Next we write  $B_n^{(-k)}$  by using the sum of  $B_n^{\overbrace{(0,\dots,0)}^{l+1}}$  ( $l \geq 1$ ) and them of  $B_n^{\overbrace{(0,\dots,0,-m)}^r}$ . First of all, we give concrete examples for  $1 \leq r \leq 3, 1 \leq k \leq 6$  such that Example 3.15.

**Example 3.16.**

We give examples for  $1 \leq r \leq 3, 1 \leq k \leq 6$ .

(i) For  $r = 1$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + B_n^{(0,-1)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 2B_n^{(0,-1)} + B_n^{(0,-2)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 3B_n^{(0,-1)} - 3B_n^{(0,-2)} + B_n^{(0,-3)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 4B_n^{(0,-1)} + 6B_n^{(0,-2)} - 4B_n^{(0,-3)} + B_n^{(0,-4)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 5B_n^{(0,-1)} - 10B_n^{(0,-2)} + 10B_n^{(0,-3)} - 5B_n^{(0,-4)} + B_n^{(0,-5)}$$

(ii) For  $r = 2$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 2B_n^{(0,0,-1)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 10B_n^{(0,0,-1)} + 2B_n^{(0,0,-2)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 34B_n^{(0,0,-1)} - 14B_n^{(0,0,-2)} + 2B_n^{(0,0,-3)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 98B_n^{(0,0,-1)} + 62B_n^{(0,0,-2)} - 18B_n^{(0,0,-3)} + 2B_n^{(0,0,-4)}$$

(iii) For  $r = 3$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 6B_n^{(0,0,0,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 36B_n^{(0,0,0,0)} + 6B_n^{(0,0,0,-1)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 150B_n^{(0,0,0,0)} - 54B_n^{(0,0,0,-1)} + 6B_n^{(0,0,0,-2)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 540B_n^{(0,0,0,0)} + 312B_n^{(0,0,0,-1)} - 72B_n^{(0,0,0,-2)} + 6B_n^{(0,0,0,-3)}$$

By Example3.16, we see the parts on coefficients except for plus or minus sign on Multi-Poly-Bernoulli numbers of the right hand. Then we obtain Pascal circles for  $r = 1$ , and from here we can be considered the case of the generalizations as follows.

**Conjecture3.17.** We will have the following relations

$$B_n^{(-k)} = \sum_{l=1}^r (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} B_n^{(0,\dots,0)} + \sum_{m=1}^{k-r} (-1)^{k-m-r} a_{k-r-1,m} B_n^{(0,\dots,0,-m)}.$$

We define that  $a_{k-r-1,m} = a_{k-r-2,m-1} + r a_{k-r-2,m}$  ( $0 \leq k-r-2$ ,  $1 \leq m \leq k-r$ ),  $a_{k-r-1,0} = r! \left\{ \begin{matrix} k \\ r \end{matrix} \right\}$ , and  $a_{k-r-1,k-r} = r!$ .

For example, for  $k = 4$  and  $r = 2$ ,

$$\begin{aligned} B_n^{(-4)} &= \sum_{l=1}^2 (-1)^{4-l} l! \left\{ \begin{matrix} 4 \\ l \end{matrix} \right\} B_n^{(0,\dots,0)} + \sum_{m=1}^2 (-1)^{2-m} a_{1,m} B_n^{(0,0,-m)} \\ &= -\left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\} B_n^{(0,0)} + 2! \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} B_n^{(0,0,0)} - a_{1,1} B_n^{(0,0,-1)} + a_{1,2} B_n^{(0,0,-2)} \\ &= -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 10B_n^{(0,0,-1)} + 2B_n^{(0,0,-2)}. \end{aligned}$$

$$(a_{1,1} = a_{0,0} + 2 \cdot a_{0,1} = 6 + 2 \cdot 2 = 10)$$

Moreover, we see the parts of coefficients on Multi-Poly-Bernoulli numbers of the identity which hold on Theorem3.14. Therefore we revisit Example3.15.

**Example3.15**(Example3.15 revisited).

We give examples of Theorem3.14 for  $1 \leq r \leq 2$ ,  $1 \leq k \leq 6$ .

(i) For  $r = 1$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(-1,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 3B_n^{(-1,0)} + 3B_n^{(-2,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 4B_n^{(-1,0)} - 6B_n^{(-2,0)} + 4B_n^{(-3,0)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 5B_n^{(-1,0)} + 10B_n^{(-2,0)} - 10B_n^{(-3,0)} + 5B_n^{(-4,0)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 6B_n^{(-1,0)} - 15B_n^{(-2,0)} + 20B_n^{(-3,0)} - 15B_n^{(-4,0)} + 6B_n^{(-5,0)}$$

(ii) For  $r = 2$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 6B_n^{(-1,0,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 24B_n^{(-1,0,0)} + 12B_n^{(-2,0,0)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 70B_n^{(-1,0,0)} - 60B_n^{(-2,0,0)} + 20B_n^{(-3,0,0)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 180B_n^{(-1,0,0)} + 210B_n^{(-2,0,0)} - 120B_n^{(-3,0,0)} + 30B_n^{(-4,0,0)}$$

Here by Example 3.15, the sum of coefficients on Multi-Poly-Bernoulli numbers are all 1 for  $r = 1$  and  $r = 2$ . From this, we can be considered the following Theorem 3.18.

**Theorem 3.18.** We have the following relations on the sum of coefficients.

$$\sum_{l=1}^r (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} + \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \left\{ \begin{matrix} k-m \\ r \end{matrix} \right\} = 1.$$

$$\begin{aligned} \text{Proof. } \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \left\{ \begin{matrix} k-m \\ r \end{matrix} \right\} &= \sum_{m=r+1}^k (-1)^{k-m} \binom{k}{m-r} r! \left\{ \begin{matrix} k-m+r \\ r \end{matrix} \right\} \\ &= \sum_{l=r+1}^k (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\}. \end{aligned}$$

Thus, the left hand of the equality equals  $\sum_{l=1}^k (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\}$ , and from the proof

of Theorem 3.12, we have  $\sum_{l=1}^k (-1)^{k-l} l! \left\{ \begin{matrix} k \\ l \end{matrix} \right\} = 1$ .

Therefore we obtain the result.  $\square$

Similary, we see the parts of coefficients on Multi-Poly-Bernoulli numbers where represents  $B_n^{(-k)}$  by using the sum of  $B_n^{(0,\dots,0)} \overset{l+1}{\text{}} (l \geq 1)$  and them of  $B_n^{(0,\dots,0,-m)} \overset{r}{\text{}} \text{}$ . But we don't find regularities, therefore we don't see the relations yet.

We give tables which show the values of  $B_n^{(k)}$  ( $-5 \leq k \leq 5, 0 \leq n \leq 7$ ) and  $B_n^{(k_1,k_2)}$  for small  $n, k_i$ .

Table 2. [1]  $B_n^{(k)}$  ( $-5 \leq k \leq 5, 0 \leq n \leq 7$ )

k \ n	0	1	2	3	4	5	6	7
-5	1	32	454	4718	41506	329462	2441314	17234438
-4	1	16	146	1066	6902	41506	237686	1315666
-3	1	8	46	230	1066	4718	20266	85310
-2	1	4	14	46	146	454	1394	4246
-1	1	2	4	8	16	32	64	128
0	1	1	1	1	1	1	1	1
1	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0
2	1	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{24}$	$\frac{7}{450}$	$\frac{1}{40}$	$-\frac{38}{2205}$	$-\frac{5}{168}$
3	1	$\frac{1}{8}$	$-\frac{11}{216}$	$-\frac{1}{288}$	$\frac{1243}{54000}$	$-\frac{49}{7200}$	$-\frac{75613}{3704400}$	$\frac{599}{35280}$
4	1	$\frac{1}{16}$	$-\frac{49}{1296}$	$\frac{41}{3456}$	$\frac{26291}{3240000}$	$-\frac{1921}{144000}$	$\frac{845233}{1555848000}$	$\frac{1048349}{59270400}$
5	1	$\frac{1}{32}$	$-\frac{179}{7776}$	$\frac{515}{41472}$	$-\frac{216383}{194400000}$	$-\frac{183781}{25920000}$	$\frac{4644828199}{653456160000}$	$\frac{153375307}{49787136000}$

Table3. [5]  $B_n^{(k_1,k_2)}$  ( $0 \leq n \leq 7, k_1, k_2$ : small values)

\ n	0	1	2	3	4	5	6	7
$B_n^{(1,1)}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{20}$	$-\frac{1}{12}$	$\frac{5}{84}$	$\frac{1}{12}$
$B_n^{(1,0)}$	1	$\frac{3}{2}$	$\frac{13}{6}$	3	$\frac{119}{30}$	5	$\frac{253}{42}$	7
$B_n^{(0,1)}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{29}{30}$	$\frac{31}{30}$	$\frac{43}{42}$	$\frac{41}{42}$	$\frac{29}{30}$
$B_n^{(0,0)}$	1	2	4	8	16	32	64	128
$B_n^{(0,-1)}$	2	6	18	54	162	486	1458	4374
$B_n^{(-1,0)}$	1	3	9	27	81	243	729	2187
$B_n^{(-1,-1)}$	2	9	39	165	687	2829	11505	46965

## References

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